Numerieke methoden in de natuurwetenschappen

Ex. 6: The vibrating string: solution with finite differences and finite elements. Points a) and b) are compulsory, point c) is facultative.

Consider a vibrating piano string fixed at both ends ($x = 0$ and $x = L$ =1 m) and under a uniform tension $T = 10^3$ N. We wish to find the allowed frequency of vibrations ω and the corresponding displacements by solving the equation:

$$\frac{d^2y(x)}{dx^2} + \frac{\omega^2\mu(x)}{T}y(x) = 0 \tag{1}$$

where $\mu(x)$ is the mass of the string per unit length. We take first a constant mass density $\mu_0 = 0.954$ g/m, and then $\mu(x) = \mu_0 + (x - \frac{L}{2})\Delta$ with Δ =0.5 g/m^2. In the first case the solutions are known and correspond to a sinusoidal motion with frequency

$$\omega = \frac{n\pi}{L}\sqrt{T/\mu} \qquad n = 1, 2, ... \tag{2}$$

Note that this problem could be also solved by the shooting technique.

a) Solve the problem by the finite difference method to find the first two eigenvalues and eigenvectors for both constant and variable mass density. Write the equations onto a grid of 11 and 21 points from 0 to L and compare the results. Find the eigenvalues by searching for the zeroes of the determinant of the resulting tridiagonal matrix and the eigenvectors by backsubstitution.

b) Find the lowest eigenvalue (the fundamental note of the piano) by the inverse power method.

c) Solve the same problem by the finite elements method. This method implys finding the eigenvalues of a generalized eigenvalue problem of the type $\mathbf{A}x = \lambda\mathbf{B}x$ which can be solved by use of the NAG libraries.