

As for [54], eqn [53] has Hermite–Gauss solutions

$$E_n(t) = H_n(2\pi t/\tau_p) \exp\{-(2\pi t/\tau_p)^2/2\} \quad [55]$$

with

$$\frac{2\pi}{\tau_p} = \left(\frac{\alpha_m L_m \omega_m^2 \Delta \omega_g^2}{2g_0 \ell} \right)^{1/4} \quad [56]$$

$$\left(1 - \frac{\alpha_i L_i}{g_0 \ell} \right) = (2n + 1) \left(\frac{2\pi}{\tau_p \Delta \omega_g} \right)^2$$

Since as a rule the gain barely exceeds the loss, this last relation can be put into a more readable form as follows:

$$\frac{g_0 \ell}{\alpha_i L_i} \left(1 - \frac{\alpha_i L_i}{g_0 \ell} \right) = \left(\frac{g_0 \ell - \alpha_i L_i}{\alpha_i L_i} \right)$$

$$= (2n + 1) \left(\frac{2\pi}{\tau_p \Delta \omega_g} \right)^2 \frac{g_0 \ell}{\alpha_i L_i}$$

$$\approx (2n + 1) \left(\frac{2\pi}{\tau_p \Delta \omega_g} \right)^2 \quad [57]$$

which represents the excess gain over the threshold gain for continuous operation, needed to sustain a mode-locked Hermite–Gauss pulse of order n : higher orders need more gain.

It can be seen that gain saturation leads to the stability of lowest-order Gaussian pulse, and to the instability of higher-order solutions. If just the right net gain is supplied to excite the fundamental solution, higher-order ones do not have enough gain to start. If the supplied gain is in excess so that higher-order solutions can also start, the amplitude of the lowest order quickly grows to the point that it saturates the gain which settles it to its steady-state value, switching off the higher-order solutions.

It is thus seen from [55] that the pulse is Gaussian in shape; furthermore, from the first of [56] it is seen

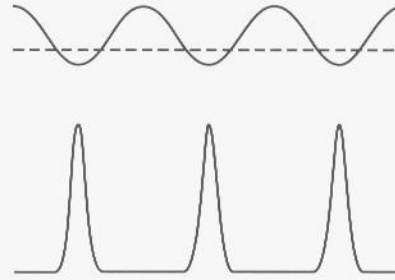


Figure 10 Gain and losses in an actively mode-locked laser, and its generated pulse train. The dotted line sets the net gain of the laser for continuous operation, and the extra, modulated loss are shown in full line above it. With periodicity of a round-trip time, a temporal window is opened in which the net gain is positive, and the pulses can go through, as shown in full line at the bottom. Pulse duration not to scale with round-trip time in the figure.

that the pulsewidth has a power dependence on both the modulation depth and the gain bandwidth.

Figure 10 illustrates qualitatively the interplay of gain and losses in an actively mode-locked laser, and its Gaussian envelope pulses.

See also: Nonlinear Optics; Optical Fibers; Optical Sum Rules and Kramers–Kronig Relations; Photon Statistics and Coherence Theory.

PACS: 42.55.Ah; 42.60.Da; 42.60.Fc; 42.60.Jf; 42.60.Pk

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Lattice Dynamics: Anharmonic Effects

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Introduction

Crystal lattice dynamics is based on the concept of phonons, that is, weakly interacting waves of atomic

(or ionic) vibrations and corresponding quasiparticles (according to a corpuscular-wave dualism of the quantum physics, any excitation wave in a system can also be described as a particle). Due to the smallness of typical atomic displacements, \bar{u} in crystals in comparison with interatomic distances d , one can pass rigorously from a problem of strongly interacting “particles” forming the crystal (atoms, ions, or molecules) to a problem of weakly

interacting “quasiparticles” (phonons). In the leading order in the smallness parameter $\eta = \bar{u}/d$, the crystal lattice dynamics and thermodynamics can be described in terms of an ideal phonon gas (harmonic approximation). For the ground state, the harmonic approximation can be justified rigorously on the basis of the adiabatic (Born–Oppenheimer) approximation of quantum mechanics due to the smallness of the ratio of the electron mass, m to the atomic (or ionic) mass M . With the increase in temperature T , the parameter η increases as well; however, due to a semi-empirical Lindemann criterion $\eta \approx 0.1$ at the melting point $T = T_m$, higher-order contributions to thermodynamic properties are usually small up to the melting temperature. This statement is true for most of the average characteristics. At the same time, for some peculiar modes, the harmonic approximation can be completely inadequate, especially in the vicinity of some structural transformations, such as ferroelectric phase transitions or martensitic transformations in metals (soft modes). It is not sufficient also for quantum crystals, such as solid He^3 and He^4 . Even in a generic case, some phenomena can be understood only beyond the picture of the ideal phonon gas, which means beyond the harmonic approximation. All these phenomena are called anharmonic. There are anharmonic effects in the crystal lattice thermodynamics (thermal expansion, temperature dependences of elastic moduli, etc.), kinetics (phonon-phonon scattering processes which are responsible for the thermal conductivity of insulators), and dynamics (phonon damping and temperature dependences of the phonon frequencies measured by the inelastic neutron scattering method).

Formulation of the Problem

In the adiabatic approximation, one can split the quantum-mechanical problem of the crystal into an electronic (a solution of the Schrödinger equation for fixed coordinates of nuclei $\{\mathbf{r}_j\}$) and a nuclear one which is the base of the theory of crystal lattice properties. Keeping in mind the smallness of the atomic displacements $\eta \ll 1$, one can expand the energy of the nuclei $V(\{\mathbf{r}_j\})$ into the displacement vectors $\mathbf{u}_j = \mathbf{r}_j - \mathbf{R}_j$, \mathbf{R}_j being the equilibrium lattice positions:

$$\begin{aligned}
 V(\{\mathbf{r}_j\}) &= V_0 + \frac{1}{2} \sum_{j_1 j_2 \alpha_1 \alpha_2} V_{j_1 j_2}^{\alpha_1 \alpha_2} u_{j_1}^{\alpha_1} u_{j_2}^{\alpha_2} \\
 &+ \frac{1}{6} \sum_{j_1 j_2 j_3 \alpha_1 \alpha_2 \alpha_3} V_{j_1 j_2 j_3}^{\alpha_1 \alpha_2 \alpha_3} u_{j_1}^{\alpha_1} u_{j_2}^{\alpha_2} u_{j_3}^{\alpha_3} + \dots, \\
 V_{j_1 \dots j_n}^{\alpha_1 \dots \alpha_n} &= \left(\frac{\partial^n V}{\partial u_{j_1}^{\alpha_1} \dots \partial u_{j_n}^{\alpha_n}} \right)_{\mathbf{u}=0}
 \end{aligned} \quad [1]$$

where α_j are Cartesian indices and the linear term is absent due to the equilibrium conditions. The harmonic approximation corresponds to taking into account only the quadratic term in the expansion [1]. Then the nuclear Hamiltonian, which is the sum of the potential energy V and the kinetic energy of nuclei, can be represented as a sum of the Hamiltonians of independent oscillators (phonons) by the transformation

$$\begin{aligned}
 \mathbf{u}_j &= \sum_{\lambda} \sqrt{\frac{\hbar}{2M_j N_0 \omega_{\lambda}}} A_{\lambda} \mathbf{e}_{\lambda} \exp(i\mathbf{q}\mathbf{R}_j), \\
 A_{\lambda} &= b_{\lambda} + b_{-\lambda}^{\dagger}
 \end{aligned} \quad [2]$$

where N_0 is the number of unit cells in the crystal, M_j is the mass of the j th nucleus, $\lambda \equiv \mathbf{q}\xi$ are the phonon quantum numbers (\mathbf{q} is the wave vector running the Brillouin zone and ξ is the polarization index, $-\lambda \equiv -\mathbf{q}\xi$), \mathbf{e}_{λ} is the polarization vector, b_{λ} and b_{λ}^{\dagger} are the annihilation and creation phonon operators. In the phonon representation, the total Hamiltonian of the crystal lattice has the form

$$\begin{aligned}
 H &= V_0 + H_0 + \sum_{k=3}^{\infty} H^{(k)}, \\
 H_0 &= \sum_{\lambda} \hbar \omega_{\lambda} \left(b_{\lambda}^{\dagger} b_{\lambda} + \frac{1}{2} \right), \\
 H^{(k)} &= \sum_{\lambda_1 \dots \lambda_k} \frac{\Phi_{\lambda_1 \dots \lambda_k}^{(k)}}{k!} A_{\lambda_1} \dots A_{\lambda_k}
 \end{aligned} \quad [3]$$

where H_0 is the Hamiltonian of the ideal phonon gas (the harmonic approximation) and the multiphonon scattering matrix elements $\Phi_{\lambda_1 \dots \lambda_k}^{(k)}$ are proportional to the k th derivatives of the potential energy and the factors $\sqrt{\hbar/2M_j \omega_{\lambda}}$. These matrix elements describe the processes of phonon–phonon interaction, such as a merging of two phonons into one or, vice versa, a decay of a phonon into two ($k=3$), scattering of two phonons into two new states ($k=4$), etc. The anharmonic effects connected with these interactions are called self-anharmonic. There is also another kind of anharmonic effect which is connected to the dependence of the phonon frequencies ω_{λ} on the interatomic distances in the Hamiltonian H_0 . These effects are called quasi-harmonic. They are characterized by the microscopic Grüneisen parameters

$$\gamma_{\lambda} = -\frac{\partial \ln \omega_{\lambda}}{\partial \ln \Omega} \quad [4]$$

where Ω is the crystal volume. For noncubic crystals, the dependence of ω_{λ} on shear deformations should also be considered. The point is that for harmonic oscillators described by the quadratic potential energy and, correspondingly, by linear interatomic forces, the frequencies are not dependent either on the

amplitudes of oscillations or on the equilibrium positions. Due to nonlinear (anharmonic) effects, there are renormalizations of the frequencies. Up to the lowest order η^2 , one should take into account the quasiharmonic effects and the self-anharmonic effects with $k = 3$ and 4.

Thermodynamics

Thermal Expansion

Thermal expansion, characterized by the coefficient,

$$\alpha_p = \frac{1}{\Omega} \left(\frac{\partial \Omega}{\partial T} \right)_p \quad [5]$$

(T is the temperature and p is the pressure) equals to zero in harmonic approximation. This can be easily proved from the Gibbs distribution for the potential energy V , which is quadratic in the atomic displacements u_j (and, moreover, if it is just an arbitrary even function of u_j). It can be calculated by using the known thermodynamic identity

$$\left(\frac{\partial \Omega}{\partial T} \right)_p = - \left(\frac{\partial S}{\partial p} \right)_T \quad [6]$$

where S is the entropy; as a consequence, due to the third law of thermodynamics, the thermal expansion coefficient should vanish at $T \rightarrow 0$. If one calculates the entropy in harmonic approximation but with the phonon frequencies dependent on the volume (quasiharmonic approximation), one can derive the Grüneisen law

$$\alpha_p = \frac{\gamma(T) C_V(T)}{\Omega B_T} \quad [7]$$

where $B_T = -\Omega(\partial p/\partial \Omega)_T$ is the isothermal bulk modulus,

$$\gamma(T) = \frac{\sum_{\lambda} \gamma_{\lambda} C_{\lambda}}{\sum_{\lambda} C_{\lambda}}, \quad C_{\lambda} = \left(\frac{\hbar \omega_{\lambda}}{k_B T} \right)^2 \frac{\exp(\hbar \omega_{\lambda}/k_B T)}{[\exp(\hbar \omega_{\lambda}/k_B T) - 1]^2} \quad [8]$$

is the macroscopic Grüneisen parameter (which is temperature independent assuming that $\gamma_{\lambda} = \text{const}$) and $C_V(T) = k_B \sum_{\lambda} C_{\lambda}$ is the constant-volume lattice heat capacity. It follows from eqn [7] that the temperature dependence of the thermal expansion coefficient at low and high temperatures is the same as for the heat capacity: $\alpha_p \sim T^3$ at $T \ll \theta_D$ (θ_D is the Debye temperature) and $\alpha_p \approx \text{const}$ at $T \geq \theta_D$.

For noncubic crystals, one should introduce parameters characterizing the anisotropy of the thermal

expansion $\alpha_i = \partial u_i / \partial T$, where u_i are different deformations, for example, for uniaxial crystals with c -axis different from a - and b -axes, one can introduce $du_1 = d \ln \Omega$ and $du_2 = d \ln(c/a)$. As a generalization of eqn [6], one can prove from the equilibrium conditions at finite temperatures

$$\alpha_i = \sum_j (B^{-1})_{ij} \left(\frac{\partial S}{\partial u_j} \right)_T \quad [9]$$

where $B_{ij} = (1/\Omega) (\partial^2 F / \partial u_i \partial u_j)$ is the matrix of isothermal elastic moduli and F is the free energy. In particular, for the uniaxial crystals

$$\begin{aligned} & \left(\frac{\partial \ln c}{\partial T} \right) \\ &= \frac{1}{3BB_{22}} \left[(B_{22} - 2B_{12}) \frac{\partial S}{\partial u_1} + (2B_{11} - B_{12}) \frac{\partial S}{\partial u_2} \right] \\ & \left(\frac{\partial \ln a}{\partial T} \right) \\ &= \frac{1}{3BB_{22}} \left[(B_{22} + B_{12}) \frac{\partial S}{\partial u_1} - (B_{11} + B_{12}) \frac{\partial S}{\partial u_2} \right] \quad [10] \end{aligned}$$

where $B = B_{11} - B_{12}^2/B_{22}$ is the bulk modulus of the uniaxial crystal.

Temperature Dependence of Elastic Moduli

Temperature dependence of the elastic moduli B_{ij} is another important anharmonic effect. This temperature dependence in quasiharmonic approximation results from the ideal phonon gas contribution to the free energy, $F_{\text{ph}} = -k_B T \sum_{\lambda} \ln[2 \sinh(\hbar \omega_{\lambda}/2k_B T)]$, and from the volume dependence of the electronic contributions to the moduli. As a result, both terms behave as $\delta B_{ij} \propto -T^4$ at $T \ll \theta_D$ and $\delta B_{ij} \propto -T$ at $T \geq \theta_D$. Normally, this contribution is negative (elastic moduli decrease with the temperature increase), but for some shear moduli in peculiar cases an opposite behavior sometimes takes place (acoustic soft modes), usually near the structural phase transitions. Empirically, for many cubic crystals the trigonal shear modulus B_{44} at the melting point is 55% of its value at zero temperature (Varshni melting criterion).

Lattice Heat Capacity at High Temperatures

In harmonic approximation, the molar constant-volume heat capacity at $T \geq \theta_D$ is independent of both the temperature and the chemical composition of the crystal: $C_V = 3R$ (Dulong–Petit law, $R \approx 8.3144 \text{ J mol}^{-1} \text{ K}^{-1}$ is the gas constant). Self-anharmonic effects lead to linear temperature dependence of the heat capacity, $\delta C_V \sim R \eta^2 \sim R(k_B T/E_{\text{coh}})$

where E_{coh} is a typical energy of the chemical bonding. These terms arise from both three-phonon and four-phonon processes (the second-order perturbation effect in $H^{(3)}$ and the first-order one in $H^{(4)}$, see eqn [3]):

$$C_V^{(3)} = \frac{4k_B^2 T}{3} \sum_{kq\xi\eta\zeta} \frac{|\Phi_{\xi k; \eta q; \zeta, -k-q}^{(3)}|^2}{\hbar^3 \omega_{\xi k} \omega_{\eta q} \omega_{\zeta, -k-q}} \quad [11]$$

$$C_V^{(4)} = -k_B^2 T \sum_{kq\xi\eta\zeta} \frac{\Phi_{\xi k; \zeta, -k; \eta q; \eta, -q}^{(4)}}{\hbar^2 \omega_{\xi k} \omega_{\eta q}} \quad [12]$$

The three-phonon contribution [11] is always positive (the growth of C_V with the increase in the temperature), whereas the four-phonon one [12] can be, in principle, of arbitrary sign. Experimental separation of the self-anharmonic contribution to the lattice heat capacity is a difficult problem since it has, in general, the same order of magnitude and temperature dependence as the difference

$$C_P - C_V = T\Omega B_T \alpha_p^2 \quad [13]$$

(C_P is the experimentally measurable heat capacity at a constant pressure) and, in metals, as the electron heat capacity.

Phonon Spectra and Damping

Temperature Dependences of Phonon Frequencies

According to classical mechanics, for a generic non-linear system, the oscillation frequencies are dependent on the oscillation amplitudes. Therefore, one can expect that anharmonic effects lead to the temperature-dependent phonon spectra, due to growth of average oscillation amplitudes with the increase in temperature. In quantum terms, the same effect can be described as an appearance of the phonon self-energy due to the phonon-phonon interaction processes. Up to the second order in the smallness parameter η , the temperature shift of the phonon frequency ω_λ is determined by the following expression:

$$\Delta\omega_\lambda = \Delta_\lambda^{(\text{qh})} + \Delta_\lambda^{(3)} + \Delta_\lambda^{(4)} \quad [14]$$

where $\Delta_\lambda^{(\text{qh})} = -\gamma_\lambda \Delta\Omega(T)/\Omega$ is the quasiharmonic contribution due to the temperature dependence of the crystal volume $\Omega(T)$, and $\Delta_\lambda^{(3)}$ and $\Delta_\lambda^{(4)}$ are the contributions of the three-phonon and four-phonon

processes, correspondingly:

$$\begin{aligned} \Delta_{\xi k}^{(3)} = & -\frac{1}{2\hbar^2} \mathcal{P} \sum_{q\eta\zeta} |\Phi_{\xi k; \eta q; \zeta, -k-q}^{(3)}|^2 \\ & \times \left(\frac{1 + N_{\eta q} + N_{\zeta, k+q}}{\omega_{\eta q} + \omega_{\zeta, k+q} + \omega_{\xi k}} + \frac{1 + N_{\eta q} + N_{\zeta, k+q}}{\omega_{\eta q} + \omega_{\zeta, k+q} - \omega_{\xi k}} \right. \\ & + \frac{N_{\eta q} - N_{\zeta, k+q}}{\omega_{\zeta, k+q} - \omega_{\eta q} - \omega_{\xi k}} \\ & \left. - \frac{N_{\eta q} - N_{\zeta, k+q}}{\omega_{\eta q} - \omega_{\zeta, k+q} - \omega_{\xi k}} \right) \quad [15] \end{aligned}$$

$$\Delta_{\xi k}^{(4)} = \frac{1}{2\hbar} \sum_{q\eta} \Phi_{\xi k; \zeta, -k; \eta q; \eta, -q}^{(4)} (1 + 2N_{\eta q}) \quad [16]$$

where \mathcal{P} is the principal value symbol and $N_\lambda = (\exp \hbar\omega_\lambda/k_B T - 1)^{-1}$ is the Planck function. At high temperatures $T \gtrsim \theta_D$, one can use the classical asymptotic $N_\lambda \simeq k_B T/\hbar\omega_\lambda$ and see that all three contributions in eqn [14] are linear in temperature. Usually, the phonon frequencies decrease with the temperature increase; a typical behavior is shown in **Figure 1** (upper panel). However, for the soft modes, $d\omega_\lambda/dT > 0$, as it is illustrated by **Figure 2**. In the framework of the perturbation theory, this behavior is connected with the contribution $\Delta_\lambda^{(4)}$.

Phonon Damping

Beyond the harmonic approximation, the phonons cannot be considered as stable quasiparticles; for example, due to the three-phonon processes, they can decay into couples of other phonons. As a result, the phonon damping arises (which can be measured experimentally as a half-width of phonon peaks in inelastic neutron scattering spectra). In the lowest-order perturbation theory, the damping, or the inverse phonon lifetime, is equal to

$$\begin{aligned} \Gamma_{\xi k} = & \frac{\pi}{2\hbar^2} \sum_{q\eta\zeta} |\Phi_{\xi k; \eta q; \zeta, -k-q}^{(3)}|^2 \{ (1 + N_{\eta q} + N_{\zeta, k+q}) \\ & \times \delta(\omega_{\eta q} + \omega_{\zeta, k+q} - \omega_{\xi k}) + (N_{\eta q} - N_{\zeta, k+q}) \\ & \times [\delta(\omega_{\zeta, k+q} - \omega_{\eta q} - \omega_{\xi k}) \\ & - \delta(\omega_{\eta q} - \omega_{\zeta, k+q} - \omega_{\xi k})] \} \quad [17] \end{aligned}$$

The delta functions in eqn [17] correspond to the energy and momentum conservation laws for the decay processes. At high temperatures $T \gtrsim \theta_D$, the damping is linear in T (see **Figure 1**, lower panel). Note that, both the damping and the frequency shift do not vanish at $T = 0$ where all $N_\lambda = 0$. These residual effects are due to quantum zero-point oscillations; they are of the order of $\sqrt{m/M}$ and small, in general (with the exception of the quantum crystals). For

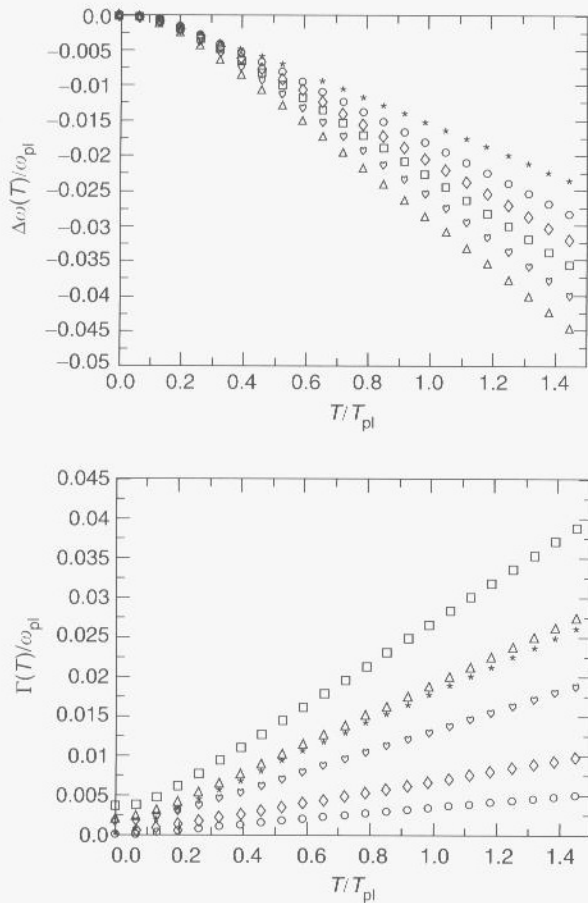


Figure 1 Temperature dependences of the frequency shifts (upper panel) and phonon damping (lower panel) for different phonon modes at the symmetric points of the Brillouin zone for face-centered cubic (f.c.c) phase of calcium. $\omega_{pl} = (4\pi Ze^2/M\Omega_0)^{1/2}$ is ionic plasma frequency, Z , M , and Ω_0 are the valency, ionic mass, and atomic volume, respectively, $T_{pl} = \hbar\omega_{pl}/k_B = 476$ K is the corresponding temperature. Asterisks and hearts label different phonon modes for L point, circles and triangles for W point, and squares and diamonds for X point. (From Katsnelson MI, Trefilov AV, Khlopin MN, and Khromov KYu (2001) *Philosophical Magazine B* 81: 1893.)

acoustic phonons with $q \rightarrow 0$, the damping [17] is linear in q , as well as in the phonon frequency; however, this is not true for the case $ql \ll 1$, where l is the phonon mean free path. For this regime, according to general hydrodynamics consideration, the damping behaves like $q^2 l$.

For strongly anharmonic modes, the phonon damping can be comparable with the frequency; phonons are not well-defined in such a situation (high-temperature body-centered cubic (b.c.c) phases of Ti and Zr can be considered as examples). The perturbation theory reviewed here can also be insufficient under resonance conditions, where the frequency ratio for different phonons with the same wave vector is close to 1:2, 1:3, etc.

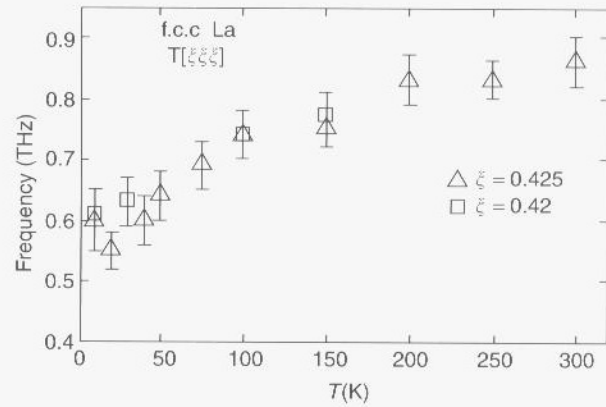


Figure 2 Temperature dependence of the phonon frequencies of the $T[\xi\xi\xi\xi]$ phonon branch in (f.c.c) lanthanum in the soft mode region. (From Stassis C, Smith GS, Harmon BN, Ho KM, and Chen Y (1985) *Physical Review B* 31: 6298.)

Transport Properties

If one neglects phonon–phonon interactions as well as phonon scattering by any defects (impurities, different isotopes, crystal surface, etc.) and by other quasiparticles (conduction electrons in metals, spin waves in magnetic crystals, etc.), the mean free path $l = \infty$ which means, in particular, infinitely large thermal conductivity κ (any nonequilibrium distribution of the phonon momenta and energies conserves for an ideal phonon gas). This means that for the case of perfect, isotopically homogeneous, large enough insulating crystals, the anharmonic effects should determine the values and temperature dependences of both l and κ ; they are connected by a simple relation:

$$\kappa = \frac{1}{3} \frac{C_V \bar{v} l}{\Omega} \quad [18]$$

where \bar{v} is a characteristic sound velocity. In principle, the same processes of phonon decay which are responsible for $\Gamma_{\xi k}$ contribute to the mean free path. However, for a continuum medium theory, any process of phonon–phonon interaction cannot lead to a finite thermal conductivity, since the momentum is conserved at any individual interaction act and any redistribution of the momenta among the phonons cannot change the energy current of the phonon gas as a whole. In crystals, Umklapp processes are possible when the momentum is conserved with the accuracy of some nonzero reciprocal lattice vector \mathbf{g} , and only these processes lead to the relaxation of the energy current and to finite κ . The energy conservation law for the Umklapp processes results in the appearance of some finite activation energy so that $l \propto \exp(\text{const} \times \theta_D/T)$ at $T \rightarrow 0$. For the very small temperatures, when the phonon mean free path is larger than the crystal size L , one should replace $l \rightarrow L$

in eqn [18] and thus $\kappa \propto C_V \propto T^3$. For high temperatures $T \geq \theta_D$, one has $\kappa \propto l \propto 1/T$.

See also: Ferroelectricity; Lattice Dynamics: Aperiodic Crystals; Lattice Dynamics: Structural Instability and Soft Modes; Lattice Dynamics: Vibrational Modes; Thermodynamic Properties, General.

PACS: 63.20.Kr; 63.20.Ry; 65.40.Ba; 65.40.De

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Lattice Dynamics: Aperiodic Crystals

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Introduction

Crystals with aperiodic structure are solids with long-range order (LRO) in the atomic positions but no translational invariance of a unit cell position. Their Fourier module consists of sharp diffraction spots, proving their LRO, which includes a long-range bond orientational order (BOO). The wave vectors Q of these diffraction spots can be represented by the sum over integer multiples of a number of rationally independent basis vectors, k , which span the Fourier module of the aperiodic crystal:

$$Q = \sum_{i=1}^n m_i k_i \quad [1]$$

In contrast to periodic crystals, where the number of rationally independent basis vectors is equal to the dimension of the crystal (and of its periodic reciprocal lattice), for aperiodic crystals n exceeds the dimension of the space, in which the crystal is embedded. The minimal necessary number of independent basis vectors defines the “rank” of the Fourier module, which is finite in the quasiperiodic case. In a space of dimension n (the n -dim space), the crystal structure, aperiodic in three-dimensional

(3D) Euclidean space, can be presented as a “periodic” structure with a unit cell, which usually has a relatively simple structure. Generally, the aperiodic structure can be obtained by an appropriate intersection of the n -dim periodic structure with the physical space, in which the aperiodic structure is observed. For quasicrystals (QCs), this intersection involves the intersection of the n -dim lattice with the physical space, which has an “irrational” slope relative to the orientation of the n -dim lattice. After a small change of this slope to a rational value, this intersection will lead to a crystal in physical space, which is periodic, but locally has a structure similar to the QCs. Such crystals are called “approximants” (to the corresponding QCs), the dynamics of which are discussed below as well.

Incommensurably modulated crystals, incommensurate composites, and QCs all belong to the aperiodic crystals. However, in some of their fundamental properties, they are different from each other. Therefore, the restriction to one or the other of them is mentioned explicitly as has just been done for QCs. Assuming that the physical (direct, external, or parallel) space, by which the n -dim space is intersected, is 3D, the remaining $(n - 3)$ -dimensional subspace is the “internal” or “perpendicular” (perp) space, which in the case of an incommensurably modulated crystal can be viewed as containing the information on the phase of the modulation function with respect to the original commensurate lattice, out of which the incommensurate phase developed. In view of the