Boltzmann equation in the presence of disorder

(Note that we consider scalar model! H is not a matrix)

self-consistent Born $\hat{\Sigma}(1,2) =$ 

$= \langle V(1)\, \hat{G}(1,2)\, V(2) \rangle_{dis}$

G which already includes Σ

no convolutions here!!!

$$\langle \ldots \rangle_{dis} = \frac{1}{V^{N_i}} \int dR_1 \int dR_2 \ldots \int dR_{N_i} \ldots$$

$R_1, R_2 \ldots R_{N_i} \leftarrow$ random positions of impurities

Impurity potential $\quad V(r) = \sum_{i=1}^{N_i} v(r - R_i)$ $\quad$ $v(r)$ – impurity profile!

let's assume $\langle V \rangle_{dis} = 0$ $\quad$ (can always be achieved by subtracting a constant (for any given N_i))

Compute first

$$\langle V(r_1)\, V(r_2) \rangle_{dis} = \sum_{i,j} \langle v(r_1 - R_i)\, v(r_2 - R_j) \rangle_{dis}$$

($= 0$ unless $i = j$! since $\langle \Sigma v(r - R_j) \rangle = 0$)

$$= \sum_{i=1}^{N_i} \langle v(r_1 - R_i)\, v(r_2 - R_i) \rangle_{dis} = \frac{N_i}{V} \int d^3R \; v(r_1 - R)\, v(r_2 - R)$$

volume

$\frac{N_i}{V} = n_{imp}$ – impurity concentration:

$$\Rightarrow \quad \langle V(r_1) V(r_2) \rangle = n_{imp} \int d^3R \; v(r_1 - R)\, v(r_2 - R) =$$

use fourier transform $\quad v(r) = \int \frac{d^3p}{(2\pi)^3} e^{ipr} v_p = \int \frac{d^3p}{(2\pi)^3} e^{-ipr} v_p^*$

since $v(r)$ – real potential

$$= n_{imp} \underbrace{\int d^3R \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} e^{ip_2 r_2 - ip_1 r_1} v_{p_2} v^*_{p_1} e^{-iR(p_2 - p_1)}}_{}$$

gives $(2\pi)^3 \delta(p_1 - p_2)$

$$= n_{imp} \int \frac{d^3p_1}{(2\pi)^3} e^{-ip_1 (r_1 - r_2)} |v_{p_1}|^2$$

$$\Sigma(1,2) = \underbrace{n_{imp} \int \frac{d^3p_1}{(2\pi)^3} e^{-ip_1(r_1 - r_2)} |v_{p_1}|^2 \, G(1,2)}_{\text{We need Wigner transform of this!}}$$

$$\Sigma(\varepsilon, p, t, R) = \int d(t_1 - t_2)\, e^{i\varepsilon(t_1 - t_2)} \underbrace{\int d^3(r_1 - r_2)\, e^{-ip(r_1 - r_2)}} \times$$

$$\times \; n_{imp} \int \frac{d^3 p_1}{(2\pi)^3} e^{-ip_1(r_1 - r_2)} \, |v_{p_1}|^2 \int \frac{d^3 p_2}{(2\pi)^3} G(t_1, t_2, p_2, R)\, e^{ip_2(r_1 - r_2)}$$

Integration over $(r_1 - r_2)$ gives $(2\pi)^3 \delta(p + p_1 - p_2)$

$=$

$\Rightarrow$ $p_1 = p_2 - p$,

$\Rightarrow$

$$\boxed{\hat{\Sigma}(\varepsilon, p, t, R) = n_{imp} \int \frac{d^3 p'}{(2\pi)^3} |v_{p-p'}|^2 \, \hat{G}(\varepsilon, p', R, t)}$$

all valid for matrices in the rotated Keldysh space!

$$\hat{\Sigma} = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix} \qquad \hat{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}$$

Let us stick to white-noise disorder for simplicity, i.e. $v_p = v_0 = \mathrm{const}(p)$

$$\Rightarrow \quad \hat{\Sigma} = \hat{\Sigma}(\varepsilon, t, R) = \underbrace{n_{imp} v_0^2}_{\text{small parameter}} \int \frac{d^3 p'}{(2\pi)^3} \hat{G}(\varepsilon, p', R, t)$$

no p dependence

we also ignore $\mathrm{Re}\,\Sigma$ (spectrum shift)

then $\Sigma^R = -i\gamma$, $\Sigma^A = i\gamma$

This can be done more accurately by solving the Excersize on page 45

Remember: equation: $\left(\partial_t - (\nabla_R U)\nabla_p + (\nabla_p \varepsilon_p)\nabla_R\right) G^{R,A} = 0$

this is solved by $\quad G^{R,A} = \dfrac{1}{\varepsilon - \varepsilon_p - U(R) \pm i\gamma}$

However, when we compute $\hat{\Sigma}$ in collision integral $[\Sigma, G]$ let us ignore the perturbation $U(R)$

i.e. disregard the driving force $\vec{F} = -\nabla_R U$ in the collision integral (standard approximation)

For Keldysh component we have:

$$\left(\partial_t - (\nabla_R U)\nabla_p + (\nabla_p \varepsilon_p)\nabla_R\right) G^K = i\left(\Sigma^K (G^R - G^A) - G^K(\Sigma^R - \Sigma^A)\right)$$

introduce distribution function by the integral

$$\underbrace{h_p}_{\text{also depends on } R \text{ and } t} = \int d\varepsilon \, \frac{G^K(\varepsilon, t, R, p)}{-2\pi i}$$

Note that $\boxed{\int d\varepsilon \, \frac{G^R - G^A}{-2\pi i} = 1}$ ← also true for any $U(R)$

↓

This correspond to the Ansatz $G^K = h_p \, (G^R - G^A)$

$h_p(R, T)$ ← no explicite energy dependence!

Now, let us see what we have for $\Sigma^K, \Sigma^R, \Sigma^A$:

$$\Sigma^K = n_{imp} v_0^2 \int \frac{d^3p'}{(2\pi)^3} \, h_{p'} \, (G^R - G^A)_{(p', R, t, \varepsilon)}$$

$$\Sigma^R - \Sigma^A = n_{imp} v_0^2 \int \frac{d^3p'}{(2\pi)^3} \, (G^R - G^A)_{(\varepsilon, p', R, t)}$$

divide by $(-2\pi i)$ & integrate over ε

$$\left(\partial_t - (\nabla_R U)\nabla_p + (\nabla_p \varepsilon_p)\nabla_R\right) h_p =$$

$$= i\, n_{imp} v_0^2 \left(\int d\varepsilon \int \frac{d^3p'}{(2\pi)^3} \left(h_{p'} (G^R_{p'} - G^A_{p'}) \frac{G^R_p - G^A_p}{-2\pi i} - (G^R_{p'} - G^A_{p'}) \frac{h_p (G^R_p - G^A_p)}{-2\pi i} \right) \right)$$

$$= 2\pi\, n_{imp} v_0^2 \int d\varepsilon \int \frac{d^3p'}{(2\pi)^3} (h_{p'} - h_p) \frac{G^R_{p'} - G^A_{p'}}{-2\pi i} \frac{G^R_p - G^A_p}{-2\pi i}$$

Consider the limit $\gamma \to 0$ $\quad \mathrm{Im}\,\Sigma \to 0 \Rightarrow \frac{G^R_p - G^A_p}{-2\pi i} \to \delta(\varepsilon - \varepsilon_p)$

In this limit we find

$$\left(\partial_t - (\nabla_R U)\nabla_p + (\nabla_p \varepsilon_p)\nabla_R\right) h_p = \underbrace{2\pi\, n_{imp} v_0^2 \int \frac{d^3p'}{(2\pi)^3} (h_{p'} - h_p)\, \delta(\varepsilon_p - \varepsilon_{p'})}_{\text{Fermi Golden rule limit}}$$

$$h_p = 1 - 2 \underbrace{f_p(R,t)}_{\text{proper distribution function}}$$

Some words on the interaction

Consider $H_{int} = g\,\psi^\dagger\psi\,\varphi$ ← bosonic field

Take a look at the self-energy  $= \Sigma$

due to interactions:

~~Define~~ Define bosonic Green's function on the contour:

$$D(1,2) = -i\langle T_{C_K}\,\varphi(1)\,\varphi(2)\rangle$$ (let it be real field)

Consider first the term $g\,\varphi(r,t)$ as a general (quantum) potential for your electronic system: (see page 15)

Remember: $G_{C_K}(1,1')\,\varphi(1')\,G_{C_K}(1',2) \Rightarrow \underline{G}(1,1')\,\tau_z\,\varphi(1')\,\underline{G}(1',2)$
in Keldysh space

in rotated space:

$$L\tau_z\,\delta G\,L^\dagger = \underbrace{L\tau_z\,\underline{G}\,L^\dagger}_{\hat G}\;\underbrace{L\tau_z\,\varphi\,\tau_z L^\dagger}_{\equiv\, L\varphi L^\dagger}\;\underbrace{L\tau_z\,\underline{G}\,L^\dagger}_{\hat G} = \hat G\,\hat\varphi\,\hat G$$

$$\hat\varphi = \begin{pmatrix}\varphi_{cl} & \varphi_q\\ \varphi_q & \varphi_{cl}\end{pmatrix}$$

So in the second order with respect to quantum potential you find:

$$\delta^{(2)}G = \hat{G}(1,2)\,\hat{\varphi}(2)\,\hat{G}(2,3)\,\hat{\varphi}(3)\,\hat{G}(3,1') = \delta^{(2)}G(1,1')$$

$$\hat{\varphi} = \begin{pmatrix} \varphi_{cl} & \varphi_q \\ \varphi_q & \varphi_{cl} \end{pmatrix} \qquad \varphi_{cl} = \frac{1}{2}\left(\varphi(t_+) + \varphi(t_-)\right) \qquad t_+ = t\,(\text{upper contour})$$

$$\varphi_q = \frac{1}{2}\left(\varphi(t_+) - \varphi(t_-)\right) \qquad t_- = t\,(\text{lower contour})$$

To compute the diagram you need to average $\langle \hat{\varphi}(2)\,\hat{\varphi}(3) \rangle$

check that $\langle \varphi_{cl}(1)\,\varphi_{cl}(2) \rangle = \frac{1}{4}\langle \underset{\text{upper}}{\varphi_+(1)\varphi_+(2)} + \underset{\text{upper}}{\varphi_+(1)}\varphi_-(2) + \varphi_-(1)\varphi_+(2) + \varphi_-(1)\varphi_-(2) \rangle$

$$= \frac{i}{4}\left(D_{11} + D_{12} + D_{21} + D_{22}\right) = \frac{i}{2}D^K(1,2)$$

$$\langle \varphi_{cl}(1)\,\varphi_q(2) \rangle = \frac{i}{4}\left(D_{11} - D_{12} + D_{21} - D_{22}\right) = \frac{i}{2}D^R(1,2)$$

$$\langle \varphi_q(1)\,\varphi_{cl}(2) \rangle = \frac{i}{4}\left(D_{11} + D_{12} - D_{21} - D_{22}\right) = \frac{i}{2}D^A(1,2)$$

$$\langle \varphi_q(1)\,\varphi_q(2) \rangle = \frac{i}{4}\left(D_{11} - D_{12} - D_{21} + D_{22}\right) = 0$$

that's the rules to compute the self energy

Homework

Use the rules at the previous page

to compute $\hat{\Sigma}(1,2) =$  $= \frac{i}{2}\sum_{i,j} \tau_i\, G(1,2)\, \tau_j\, D_{ij}(1,2)$

$i = 0,1, \quad j = 0,1$

$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$D_{00} = D^K, \quad D_{01} = D^R, \quad D_{10} = D^A$

$D_{11} = 0$

Go to Wigner coordinates for Σ

Derive corresponding <u>collision integral</u> !!!

Consider first „bare" Coulomb interaction

$D_0^R(q,\omega) = D_0^A(q) = -\frac{2\pi e^2}{q}$ ← in 2D

Remember that „bare" Coulomb is instantaneous:

$D_0(1,2) = -\frac{e^2}{|r_1 - r_2|}\,\delta(t_1 - t_2)$

$D_0^K = 0 = n(\omega)\left(D^R(q,\omega) - D^A(q,\omega)\right)$

Think of the RPA series:

↙ In RPA finite D^K will emerge!

and how to sum them up